Box-Ball Systems

Atsuo Kuniba (Univ. Tokyo)

Program on Classical, Quantum and Probabilistic Integrable Systems
- Novel Interactions and Applications @CMSA, Harvard, Boston

2 April 2025

Box-ball systems (BBS) brief chronicle

1990 First example by Takahashi-Satsuma

1996 Ultradiscretization (UD)
Classical integrability

1999 Connection to crystal base theory
Quantum (Yang-Baxter) integrability,
Quantum group theoretical generalizations

2006 Combinatorial Bethe ansatz
Action-angle variables, KKR bijection, Fermionic formulas,
Solution of initial value problem

2018 Randomized BBS
Generalized Gibbs ensemble, Thermodynamic Bethe ansatz,
Limit shape of conserved Young diagrams

2019- Generalized hydrodynamics
Speed equation, Riemann problem,
Current fluctuations and large deviations

n-color Box-ball system (BBS)

$n = 3$ example.

$$\cdots 00000000\color{red}{33211}\color{black}{0000000000000000000000000} \cdots$$
$$\cdots 0000000000000\color{red}{33211}\color{black}{00000000000000000000} \cdots$$
$$\cdots 000000000000000000\color{red}{33211}\color{black}{000000000000000} \cdots$$

0 =empty box, $\quad 1, 2, 3$ = balls with colors

- time evolution = (move 1) · (move 2) · (move 3)

(move i) · Pick the leftmost ball with color i and move it to the nearest right empty box.

· Do the same for the other color i balls.

- soliton=consecutive balls $i_1 \ldots i_a$ with color $i_1 \geq \cdots \geq i_a \geq 1$.
- velocity=amplitude.

- **Collisions of 2 solitons**

$$\cdots 000\color{red}{33211}\color{black}{0000}\color{red}{322}\color{black}{00000000000000000000000} \cdots$$
$$\cdots 00000000\color{red}{33211}\color{black}{00}\color{red}{322}\color{black}{00000000000000000000} \cdots$$
$$\cdots 0000000000000\color{red}{33211322}\color{black}{00000000000000000} \cdots$$
$$\cdots 000000000000000000\color{red}{21133322}\color{black}{000000000000} \cdots$$
$$\cdots 000000000000000000000\color{red}{211}\color{black}{00}\color{red}{33322}\color{black}{0000000} \cdots$$
$$\cdots 000000000000000000000000\color{red}{211}\color{black}{0000}\color{red}{33322}\color{black}{00} \cdots$$

- **Amplitudes are individually conserved.**

- **Two body scattering:**

  Exchange of internal labels (colors) like quarks in hadrons

  Phase shift

Collision of 3 solitons

$\cdots$ 00321003100002000000000000000 $\cdots$
$\cdots$ 00000321031000200000000000000 $\cdots$
$\cdots$ 00000000320311020000000000000 $\cdots$
$\cdots$ 00000000003200312100000000000 $\cdots$
$\cdots$ 00000000000032001032100000000 $\cdots$
$\cdots$ 00000000000000320100032100000 $\cdots$
$\cdots$ 00000000000000003021000032100 $\cdots$

$\cdots$ 00321000031002000000000000000 $\cdots$
$\cdots$ 00000321000310200000000000000 $\cdots$
$\cdots$ 00000000321003120000000000000 $\cdots$
$\cdots$ 00000000000321013200000000000 $\cdots$
$\cdots$ 00000000000000302132100000000 $\cdots$
$\cdots$ 00000000000000030021032100000 $\cdots$
$\cdots$ 00000000000000003000210032100 $\cdots$

Yang-Baxter relation is valid.

(Solitons in final state are independent of the order of collisions)

Double (classical and quantum) origin of integrability

(1) Ultra-Discretization (UD) of soliton equations

- Key formula

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) + \exp(\frac{b}{\varepsilon}) \right) = \max(a, b)$$

$$\lim_{\varepsilon \to +0} \varepsilon \log \left(\exp(\frac{a}{\varepsilon}) \times \exp(\frac{b}{\varepsilon}) \right) = a + b$$

$$(+, \times) \longrightarrow (\max, +)$$

keeps distributive law:

$$AB + AC = A(B + C) \to \max(a + b, a + c) = a + \max(b, c)$$

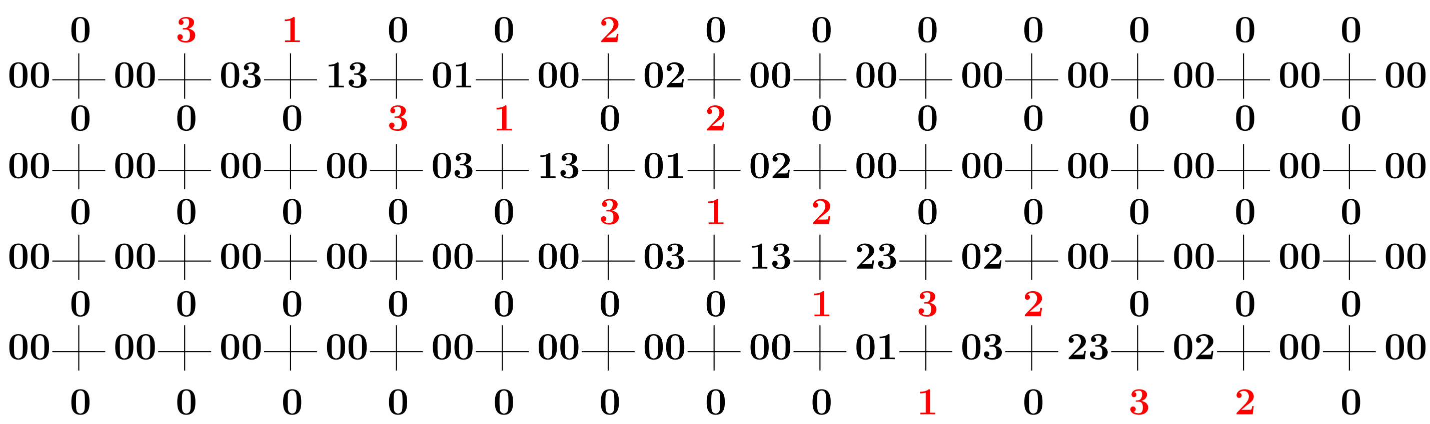
- UD of a discrete KdV equation gives an evolution equation of the $n = 1$ BBS (1996).

(2) Solvable lattice model at “ Temperature 0 ”

Time evolution pattern

$$
\begin{array}{c}
\cdots 0310020000000 \cdots \\
\cdots 0003102000000 \cdots \\
\cdots 0000031200000 \cdots \\
\cdots 0000000132000 \cdots \\
\cdots 0000000010320 \cdots
\end{array}
$$

emerges from a configuration of a 2D lattice model in statistical mechanics



by forgetting the hidden variables on the horizontal edges called carrier.

- n-color box-ball system

  = 2D solvable vertex model associated with quantum group

  $U_q(\widehat{sl}_{n+1})$ at $q = 0$ ($q \sim$ temperature)

- Row transfer matrix at $q = 0$

  = deterministic map (defined by the unique configuration surviving at $q = 0$)

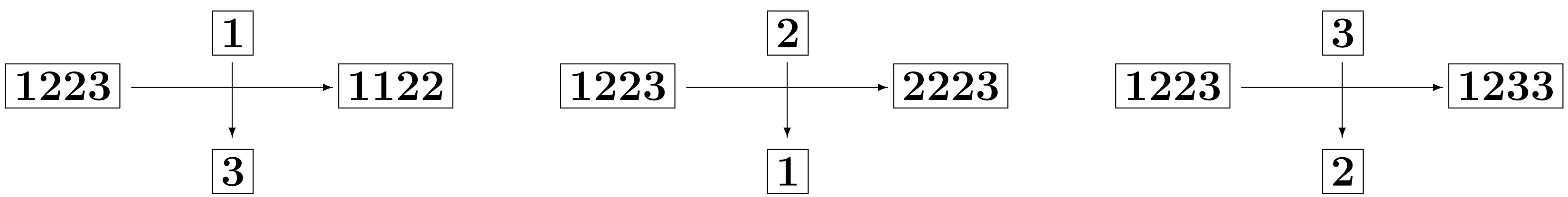
  = time evolution of box-ball system (forming a commuting family $T_1, T_2, \ldots T_\infty$)

- Proper formulation uses *crystal base theory* (theory of quantum group at $q = 0$).

  Local dynamics

  = Interaction of a carrier and a local box (pick-up/down rule of balls)

  = Combinatorial R (= Quantum R matrix at $q = 0$)



Some outcomes from such insight

- $\exists$ Integrable cellular automata with quantum group symmetry.

Example: $D_5^{(1)} = \widehat{\mathrm{so}}_{10}$ -automaton

$$\cdots 000\bar{2}\bar{4}2110000\bar{1}\bar{1}\bar{4}00000000000000000000000000000000 \cdots$$
$$\cdots 00000000\bar{2}\bar{4}21100\bar{1}\bar{1}\bar{4}00000000000000000000000000000 \cdots$$
$$\cdots 0000000000000\bar{2}\bar{4}211\bar{1}\bar{1}\bar{4}00000000000000000000000000 \cdots$$
$$\cdots 000000000000000000\bar{2}\bar{4}2\bar{0}\bar{0}\bar{4}00000000000000000000000 \cdots$$
$$\cdots 000000000000000000000003\bar{0}\bar{0}\bar{3}\bar{4}\bar{4}000000000000000000 \cdots$$
$$\cdots 00000000000000000000000000311\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}0000000000000 \cdots$$
$$\cdots 0000000000000000000000000000031100\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}00000000 \cdots$$
$$\cdots 000000000000000000000000000000003110000\bar{1}\bar{1}\bar{3}\bar{4}\bar{4}000 \cdots$$

- Particles and antiparticles undergo pair-creations/annihilations.

- n-color BBS $= \widehat{sl}_{n+1}$-automaton $= \widehat{\mathrm{so}}_{2n+2}$-automaton in antipaticle-free sector.

- Solitons & scattering rule: most naturally described in terms of crystal theory.

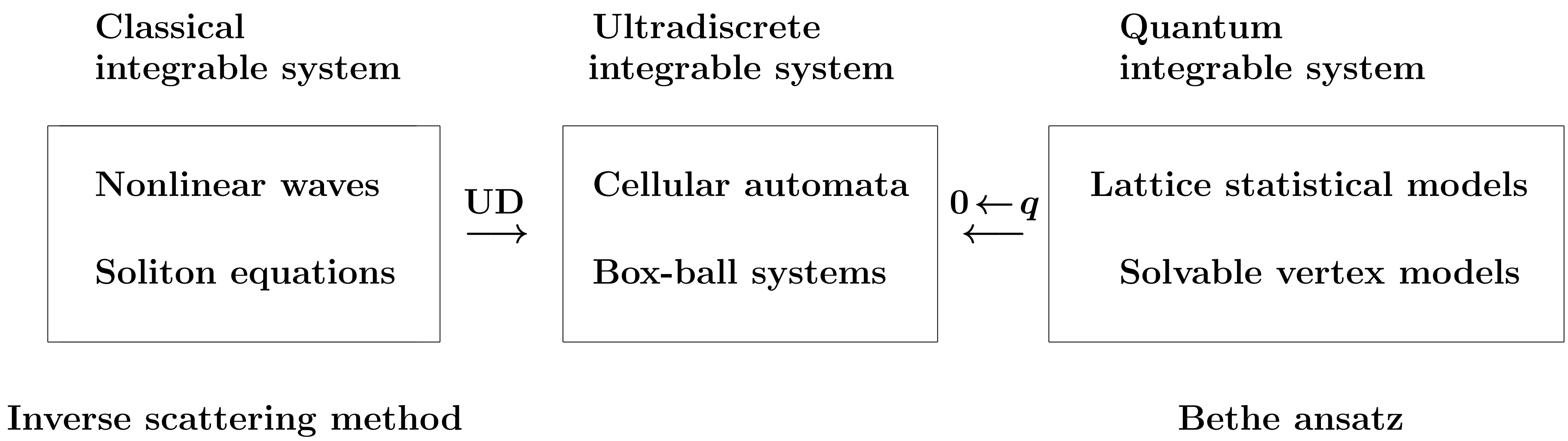
  Scattering rule of $\mathfrak{g}_n$-automaton $=$ Affine Combinatorial R of $\mathfrak{g}_{n-1}$.

Box-ball system with reflecting end

.. $5\bar{6}$ $4\bar{6}$ $3\bar{6}$ $3\bar{6}$ $2\bar{6}$ .. .. $5\bar{6}$ $4\bar{6}$ $2\bar{6}$ .. ..
.. .. $5\bar{6}$ $4\bar{6}$ $3\bar{6}$ $3\bar{6}$ .. $5\bar{6}$ $4\bar{6}$ $2\bar{6}$ $2\bar{6}$ .. ..
.. .. $5\bar{6}$ $4\bar{6}$ $3\bar{6}$ .. .. $5\bar{6}$ $4\bar{6}$ $3\bar{6}$ $2\bar{6}$ $2\bar{6}$ .. .. ..
.. .. $5\bar{6}$ $4\bar{6}$ $3\bar{6}$ .. .. $5\bar{6}$ $4\bar{2}$ $2\bar{3}$
.. .. $5\bar{2}$ $4\bar{2}$ $2\bar{3}$ $1\bar{4}$ $1\bar{5}$ .. ..
.. .. $1\bar{2}$ $1\bar{2}$ $1\bar{3}$ $1\bar{3}$ $1\bar{4}$ .. .. .. $4\bar{6}$ $3\bar{6}$ $2\bar{6}$ ..
.. .. $1\bar{2}$ $1\bar{2}$ $1\bar{3}$ $1\bar{3}$ $1\bar{4}$ .. .. $1\bar{2}$ $4\bar{3}$
.. $1\bar{2}$ $1\bar{2}$ $1\bar{3}$ $1\bar{3}$ $1\bar{4}$ .. .. $1\bar{2}$ $1\bar{3}$ $1\bar{5}$ .. ..

.. $5\bar{6}$ $4\bar{6}$ $3\bar{6}$ $3\bar{6}$ $2\bar{6}$ .. .. $5\bar{6}$ $4\bar{6}$ $2\bar{6}$ .. .. ..
.. .. $5\bar{6}$ $4\bar{6}$ $3\bar{6}$ $3\bar{6}$ $2\bar{6}$ .. .. $5\bar{6}$ $4\bar{6}$ $2\bar{6}$
.. .. $5\bar{6}$ $4\bar{6}$ $3\bar{6}$ $3\bar{6}$ $2\bar{6}$ .. .. $1\bar{3}$ $1\bar{4}$ $1\bar{5}$
.. .. $5\bar{6}$ $4\bar{2}$ $3\bar{4}$ $2\bar{5}$ $2\bar{6}$ .. ..
.. .. $1\bar{2}$ $1\bar{3}$ $1\bar{4}$ .. .. .. $1\bar{2}$ $4\bar{2}$ $2\bar{3}$
.. .. $1\bar{2}$ $1\bar{3}$ $1\bar{4}$ .. $1\bar{2}$ $1\bar{2}$ $1\bar{3}$ $1\bar{3}$ $1\bar{5}$ .. .. ..
.. .. $1\bar{2}$ $1\bar{2}$ $1\bar{3}$ $1\bar{3}$ $1\bar{4}$ .. $1\bar{2}$ $1\bar{3}$ $1\bar{5}$ .. ..
.. $1\bar{2}$ $1\bar{2}$ $1\bar{3}$ $1\bar{3}$ $1\bar{4}$ .. .. .. $1\bar{2}$ $1\bar{3}$ $1\bar{5}$ .. ..

Boundary reflections of two solitons satisfy the reflection equation $RKRK = KRKR$.


Classical
integrable system
Ultradiscrete
integrable system
Quantum
integrable system
Nonlinear waves
Soliton equations
UD
Cellular automata
Box-ball systems
0 ← q
Lattice statistical models
Solvable vertex models
Inverse scattering method
Bethe ansatz

Classical integrable system		Ultradiscrete integrable system		Quantum integrable system
Nonlinear waves Soliton equations	$\overset{\text{UD}}{\longrightarrow}$	Cellular automata Box-ball systems	$\overset{0 \leftarrow q}{\longleftarrow}$	Lattice statistical models Solvable vertex models
Inverse scattering method		KKR bijection		Bethe ansatz

• Kerov-Kirillov-Reshetikhin (KKR) bijection (1986) asserts "formal completeness" of the hypothetical string solutions to the Bethe equation at a combinatorial level.

It leads to a "Fermionic formula" for Kostka polynomials and their generalizations. (The simplest spin $\frac{1}{2}$ case with $q = 1$ dates back to [Bethe 1931]).

Classical integrable system		Ultradiscrete integrable system		Quantum integrable system
Nonlinear waves Soliton equations	UD $\longrightarrow$	Cellular automata Box-ball systems	$0 \leftarrow q$ $\longleftarrow$	Lattice statistical models Solvable vertex models
Inverse scattering method		KKR bijection		Bethe ansatz

- Kerov-Kirillov-Reshetikhin (KKR) bijection (1986) asserts "formal completeness" of the hypothetical string solutions to the Bethe equation at a combinatorial level.

  It leads to a "Fermionic formula" for Kostka polynomials and their generalizations. (The simplest spin $\frac{1}{2}$ case with $q = 1$ dates back to [Bethe 1931]).

- Its remarkable connection to BBS was discovered in 2002.

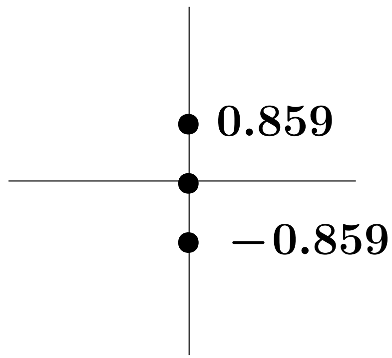
  Quasi-particles consisting of the Fermionic formula = BBS solitons!

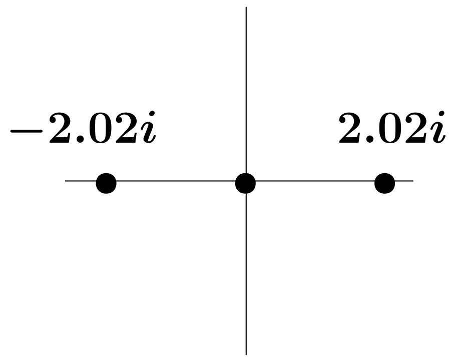
- **Example. Spin $\frac{1}{2}$ periodic Heisenberg chain**

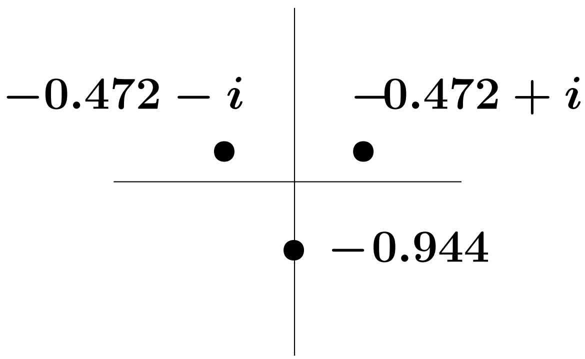
$$\boldsymbol{H} = \sum_{k=1}^{L} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z - 1)$$

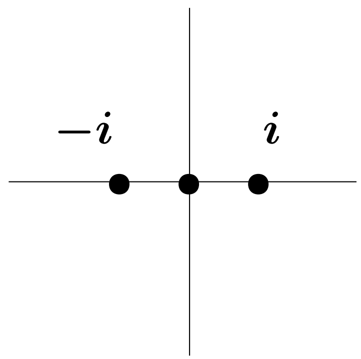
For $L = 6$ sites in 3 down-spin sector, the Bethe equation reads

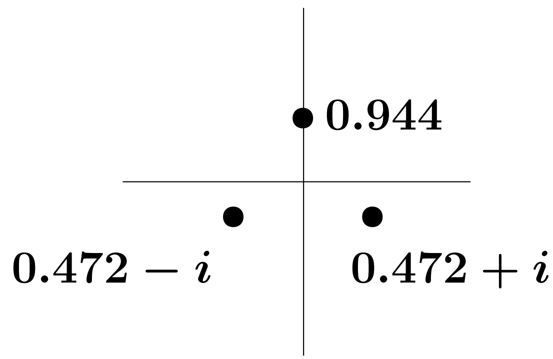
$$\left(\frac{u_1 + i}{u_1 - i}\right)^6 = \frac{(u_1 - u_2 + 2i)(u_1 - u_3 + 2i)}{(u_1 - u_2 - 2i)(u_1 - u_3 - 2i)},$$
$$\left(\frac{u_2 + i}{u_2 - i}\right)^6 = \frac{(u_2 - u_1 + 2i)(u_2 - u_3 + 2i)}{(u_2 - u_1 - 2i)(u_2 - u_3 - 2i)},$$
$$\left(\frac{u_3 + i}{u_3 - i}\right)^6 = \frac{(u_3 - u_1 + 2i)(u_3 - u_2 + 2i)}{(u_3 - u_1 - 2i)(u_3 - u_2 - 2i)}.$$

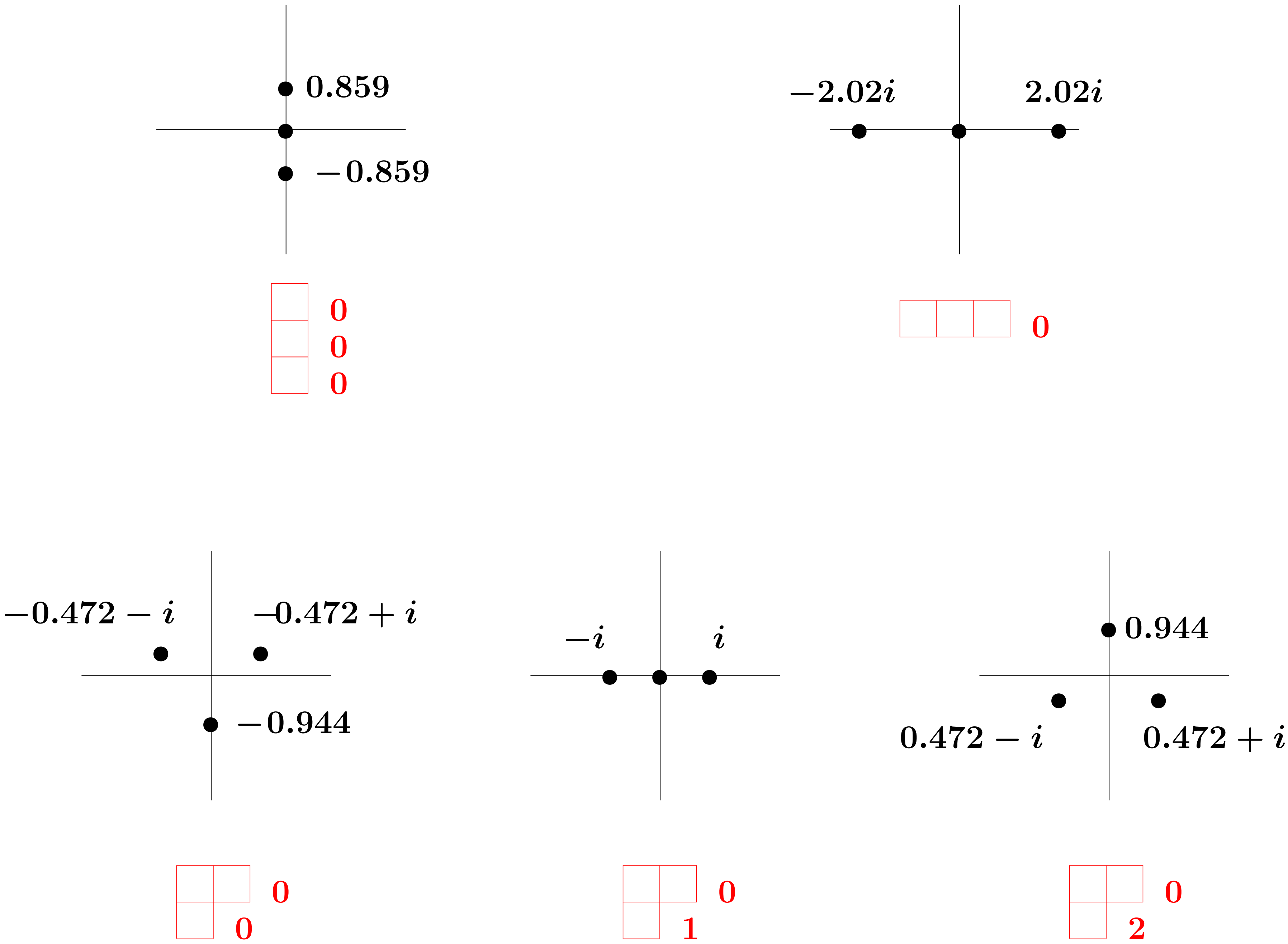

0.859
−0.859

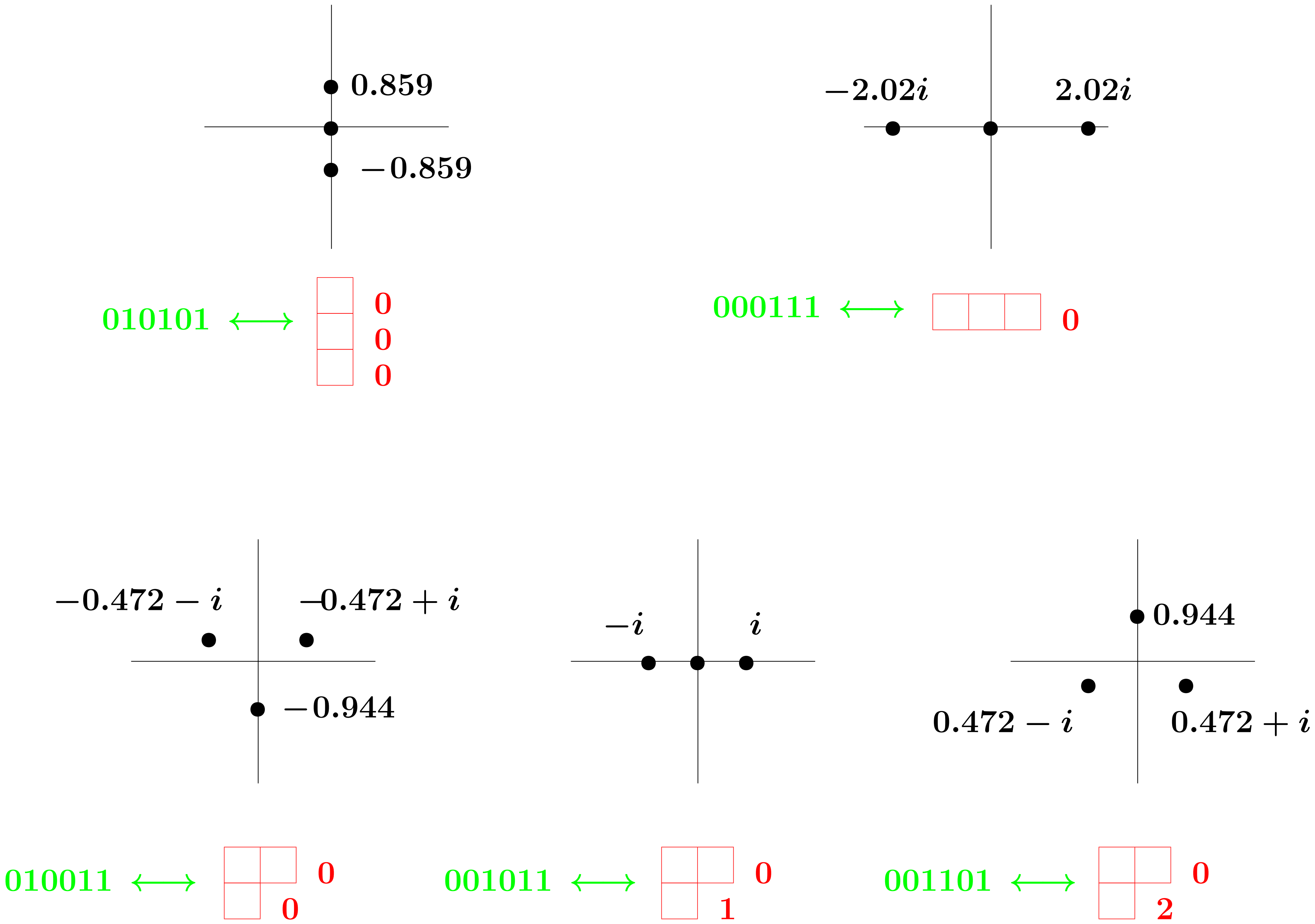

−2.02i
2.02i


−0.472 − i
−0.472 + i
−0.944


−i
i


0.944
0.472 − i
0.472 + i


0.859
−0.859
0
0
0
−2.02i
2.02i
0
−0.472 − i
−0.472 + i
−0.944
0
0
−i
i
0
1
0.944
0.472 − i
0.472 + i
0
2



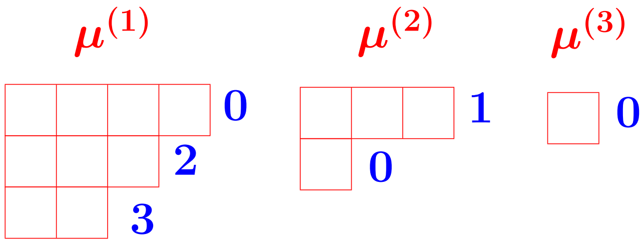
0.859
−0.859
010101 ⟷
0
0
0
−2.02i
2.02i
000111 ⟷
0
−0.472 − i
−0.472 + i
−0.944
010011 ⟷
0
0
−i
i
001011 ⟷
0
1
0.944
0.472 − i
0.472 + i
001101 ⟷
0
2

KKR bijection for sl_{n+1}

$$\{\text{highest states}\} \overset{1:1}{\longleftrightarrow} \{\text{rigged configurations}\}$$

$n = 3$ example

0000111021132200000 $\longleftrightarrow$



"Bethe vectors" "Bethe roots"

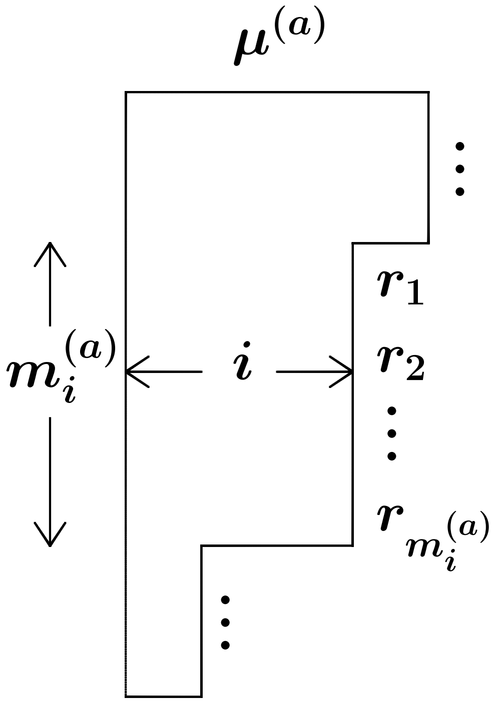
- highest states $= i_1 i_2 \ldots i_L$ $(0 \le i_k \le n)$ satisfying the highest condition:

$$\#_0\{i_1,\ldots,i_k\} \ge \#_1\{i_1,\ldots,i_k\} \ge \cdots \ge \#_n\{i_1,\ldots,i_k\} \quad (\forall k)$$

- rigged configuration: $((\mu^{(1)}, r^{(1)}),\ldots,(\mu^{(n)}, r^{(n)}))$

$\mu^{(1)},\ldots,\mu^{(n)}$: configuration $=$ n-tuple of Young diagrams
$r^{(1)},\ldots,r^{(n)}$: rigging $=$ integers assigned to each row

$+$ selection rule (next page)



$$m_i^{(a)} = \#(\text{length } i \text{ rows in } \mu^{(a)})$$

$$0 \le r_1 \le \cdots \le r_{m_i^{(a)}} \le h_i^{(a)}$$

$\cdots$ "Fermionic" selection rule

$$h_i^{(a)} = L\delta_{a,1} - \sum_{b=1}^{n} C_{ab} \sum_{j \ge 1} \min(i,j) m_j^{(b)}$$

$\cdots$ vacancy for *holes*

(C_{ab}) $\cdots$ Cartan matrix of sl_{n+1}

$$\# \text{ of rigging choices for a fixed configuration} = \prod_{a=1}^{n} \prod_{i \ge 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}$$

This is an sl_{n+1} generalization of Bethe's formula for # of string solutions (1931), which yields the so-called Fermionic character formula for KR modules.

hat also eine Möglichkeit weniger, die des letzten Komplexes von n Wellen, λ_{q_n}, kann schließlich nur noch

$$Q_n' - (q_n - 1) = Q_n + 1$$

verschiedene Werte annehmen, wo

$$Q_n (N, q_1 q_2 \ldots) = N - 2 \sum_{p<n} p\, q_p - 2 \sum_{p \geqq n} n\, q_p . \tag{44}$$

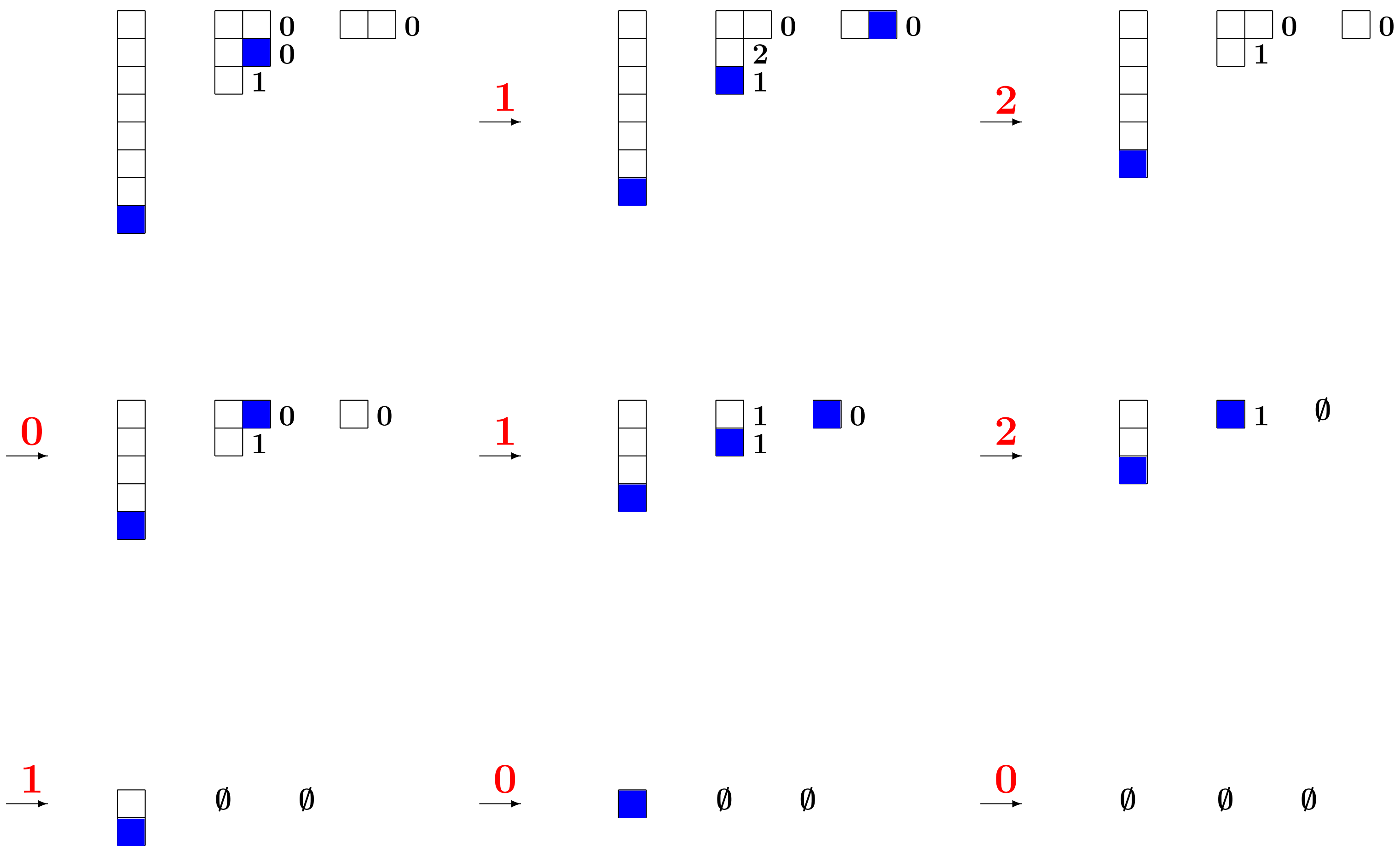
Schließlich ist zu berücksichtigen, daß Vertauschung der λ der verschiedenen Wellenkomplexe mit gleicher Anzahl n von Wellen nicht zu neuen Lösungen führt. Die gesamte Zahl unserer Lösungen wird somit

$$z (N, q_1 q_2 \ldots) = \prod_{n=1}^{\infty} \frac{(Q_n + q_n) \ldots (Q_n + 1)}{q_n!} = \prod_n \binom{Q_n + q_n}{q_n}, \tag{45}$$

wo die Q_n durch (44) definiert sind.

8. Wir werden nun nachweisen, daß wir die richtige Anzahl Lösungen erhalten haben.

Example of KKR algorithm



Top left rigged configuration $\xmapsto{\text{KKR}}$ **00121021**

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000
$t = 1$: 000000001111000022103200000000000000000000000000
$t = 2$: 000000000000111100022132000000000000000000000000
$t = 3$: 000000000000000011110021322000000000000000000000
$t = 4$: 000000000000000000001110211322000000000000000000
$t = 5$: 000000000000000000000001100211322100000000000000
$t = 6$: 000000000000000000000000011000211032210000000000
$t = 7$: 000000000000000000000000000110000211003221000000

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000

$t = 1$: 000000001111000022103200000000000000000000000000

$t = 2$: 000000000000111100022132000000000000000000000000

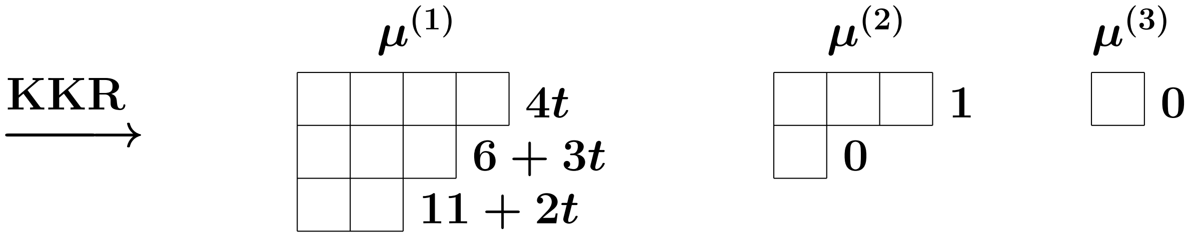
$t = 3$: 000000000000000011110021322000000000000000000000

$t = 4$: 000000000000000000001110211322000000000000000000

$t = 5$: 000000000000000000000001100211322100000000000000

$t = 6$: 000000000000000000000000011000211032210000000000

$t = 7$: 000000000000000000000000000110000211003221000000



How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000
$t = 1$: 000000001111000022103200000000000000000000000000
$t = 2$: 000000000000111100022132000000000000000000000000
$t = 3$: 000000000000000011110021322000000000000000000000
$t = 4$: 000000000000000000001110211322000000000000000000
$t = 5$: 000000000000000000000001100211322100000000000000
$t = 6$: 000000000000000000000000011000211032210000000000
$t = 7$: 000000000000000000000000000110000211003221000000

$\xrightarrow{\text{KKR}}$

$\mu^{(1)}$		$\mu^{(2)}$		$\mu^{(3)}$	
row of 4 boxes	$4t$	row of 3 boxes	1	row of 1 box	0
row of 3 boxes	$6 + 3t$	row of 1 box	0		
row of 2 boxes	$11 + 2t$				

- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000

$t = 1$: 000000001111000022103200000000000000000000000000

$t = 2$: 000000000000111100022132000000000000000000000000

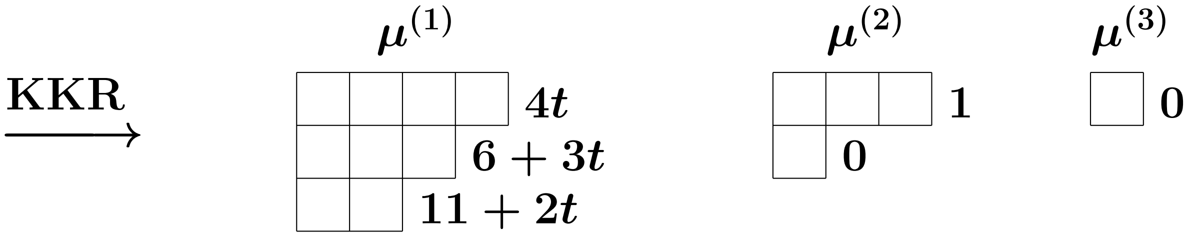
$t = 3$: 000000000000000011110021322000000000000000000000

$t = 4$: 000000000000000000001110211322000000000000000000

$t = 5$: 000000000000000000000001100211322100000000000000

$t = 6$: 000000000000000000000000011000211032210000000000

$t = 7$: 000000000000000000000000000110000211003221000000



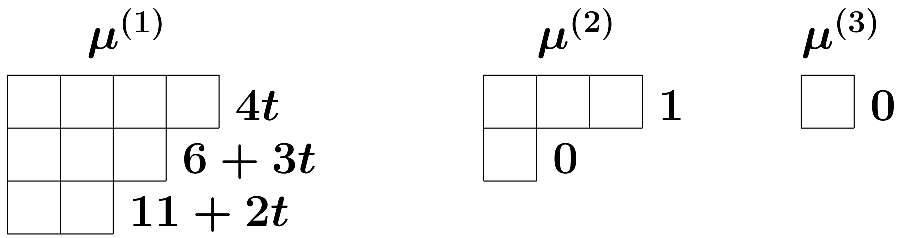
- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

Rigged configuration = action angle variable of BBS!

How does the BBS dynamics look like in terms of rigged configurations ?

$t = 0$: 000011110000022100320000000000000000000000000000
$t = 1$: 000000001111000022103200000000000000000000000000
$t = 2$: 000000000000111100022132000000000000000000000000
$t = 3$: 000000000000000011110021322000000000000000000000
$t = 4$: 000000000000000000001110211322000000000000000000
$t = 5$: 000000000000000000000001100211322100000000000000
$t = 6$: 000000000000000000000000011000211032210000000000
$t = 7$: 000000000000000000000000000110000211003221000000

$\xrightarrow{\text{KKR}}$



- Configuration is conserved (action variable)
- Rigging flows linearly (angle variable)
- KKR bijection linearizes the dynamics (direct/inverse scattering map)

Rigged configuration = action angle variable of BBS!

$\mu^{(1)}$ = list of amplitude of solitons (Soliton/String correspondence)

$(\mu^{(1)}, \ldots, \mu^{(n)})$ will be called a soliton content.

Randomized box-ball system

BBS state		Soliton content
$i_1 i_2 \dots i_L 00000 \dots$	$\xrightarrow{\text{KKR}}$	$(\mu^{(1)}, \dots, \mu^{(n)})$

Randomize $i_1 i_2 \dots i_L$ by introducing the i.i.d. measure on the set of states:

$$\text{Prob(local state} = i) = p_i \qquad (p_0 + \cdots + p_n = 1).$$

Randomized box-ball system

BBS state **Soliton content**

$$i_1 i_2 \ldots i_L 00000 \ldots \quad \overset{\mathrm{KKR}}{\longmapsto} \quad (\mu^{(1)}, \ldots, \mu^{(n)})$$

Randomize $i_1 i_2 \ldots i_L$ by introducing the i.i.d. measure on the set of states:

$$\mathrm{Prob}(\text{local state} = i) = p_i \qquad (p_0 + \cdots + p_n = 1).$$

Limit shape Problem

Determine the scaling form of the most probable $(\mu^{(1)}, \ldots, \mu^{(n)})$ when $L \to \infty$.

Randomized box-ball system

BBS state		Soliton content
$i_1 i_2 \dots i_L 00000 \dots$	$\overset{\text{KKR}}{\longmapsto}$	$(\mu^{(1)}, \dots, \mu^{(n)})$

Randomize $i_1 i_2 \dots i_L$ by introducing the i.i.d. measure on the set of states:

$$\text{Prob}(\text{local state} = i) = p_i \qquad (p_0 + \cdots + p_n = 1).$$

Limit shape Problem

Determine the scaling form of the most probable $(\mu^{(1)}, \dots, \mu^{(n)})$ when $L \to \infty$.

This can be done by TBA minimizing the free energy F associated with

$$\text{Prob}(\mu^{(1)}, \dots, \mu^{(n)}) \propto \mathrm{e}^{-\beta_1 |\mu^{(1)}| - \cdots - \beta_n |\mu^{(n)}|} \prod_{a=1}^{n} \prod_{i \geq 1} \binom{h_i^{(a)} + m_i^{(a)}}{m_i^{(a)}},$$

$$\mathrm{e}^{\beta_a} := p_{a-1}/p_a,$$

Introduce the scaled string and hole densities $\rho_i^{(a)}, \sigma_i^{(a)}$ by

$$m_i^{(a)} \simeq L\rho_i^{(a)}, \quad h_i^{(a)} \simeq L\sigma_i^{(a)}, \quad \sigma_i^{(a)} = \delta_{a,1} - \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i,j) \rho_j^{(b)},$$

Assume $p_0 \geq \cdots \geq p_n$ in the rest.

The condition $\frac{\delta F}{\delta \rho_i^{(a)}} = 0$ leads to the TBA equation

$$-i\beta_a + \log(1 + Y_i^{(a)}) = \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i,j) \log(1 + (Y_j^{(b)})^{-1})$$

in terms of $Y_i^{(a)} = \frac{\sigma_i^{(a)}}{\rho_i^{(a)}}$ with the boundary condition $\lim_{i \to \infty} \frac{1 + Y_{i+1}^{(a)}}{1 + Y_i^{(a)}} = e^{\beta_a}$.

This is equivalent to the (constant) Y-system

$$\left(Y_i^{(a)}\right)^2 = \frac{(1 + Y_{i-1}^{(a)})(1 + Y_{i+1}^{(a)})}{\left(1 + (Y_i^{(a-1)})^{-1}\right)\left(1 + (Y_i^{(a+1)})^{-1}\right)}$$

The condition $\frac{\delta F}{\delta \rho_i^{(a)}} = 0$ leads to the TBA equation

$$-i\beta_a + \log(1 + Y_i^{(a)}) = \sum_{b=1}^{n} C_{ab} \sum_{j \geq 1} \min(i,j) \log(1 + (Y_j^{(b)})^{-1})$$

in terms of $Y_i^{(a)} = \frac{\sigma_i^{(a)}}{\rho_i^{(a)}}$ with the boundary condition $\lim_{i \to \infty} \frac{1 + Y_{i+1}^{(a)}}{1 + Y_i^{(a)}} = e^{\beta_a}$.

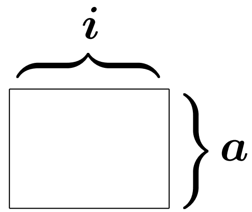
This is equivalent to the (constant) Y-system

$$\left(Y_i^{(a)}\right)^2 = \frac{(1 + Y_{i-1}^{(a)})(1 + Y_{i+1}^{(a)})}{\left(1 + (Y_i^{(a-1)})^{-1}\right)\left(1 + (Y_i^{(a+1)})^{-1}\right)}$$

Solution (Rare case for which an exact formula can be given)

$$Y_i^{(a)} = \frac{Q_{i-1}^{(a)} Q_{i+1}^{(a)}}{Q_i^{(a-1)} Q_i^{(a+1)}},$$

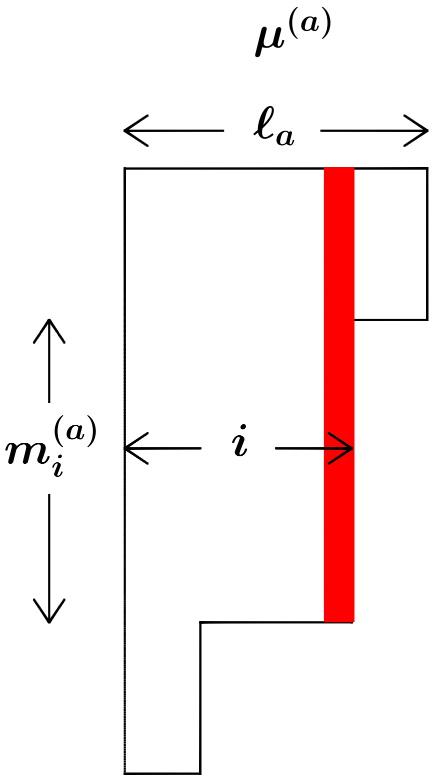
$$Q_i^{(a)} = Q_i^{(a)}(p_0, \ldots, p_n) = \frac{\det(p_k^{\lambda_j + n - j})_{j,k=0}^n}{\det(p_k^{n-j})_{j,k=0}^n} \qquad \left((\lambda_0, \ldots, \lambda_n) = (\overbrace{i \ldots i}^{a}\, \overbrace{0, \ldots, 0}^{n+1-a}) \right)$$

= Schur function for $a \times i$ rectangular Young diagram 

Result. The limit shape of soliton content $(\mu^{(1)}, \ldots, \mu^{(n)})$ is given by

$$\eta_i^{(a)} := \lim_{L\to\infty} \frac{1}{L}(\text{Length of the } i \text{ th column of } \mu^{(a)}) = \frac{Q_{i-1}^{(a-1)} Q_i^{(a+1)}}{Q_i^{(a)} Q_{i-1}^{(a)} Q_1^{(1)}}$$

$$\text{width } \ell_a \text{ of } \mu^{(a)} \simeq \frac{\log L}{\log \frac{p_{a-1}}{p_a}} \quad (L \to \infty \text{ if } p_0 > \cdots > p_n)$$

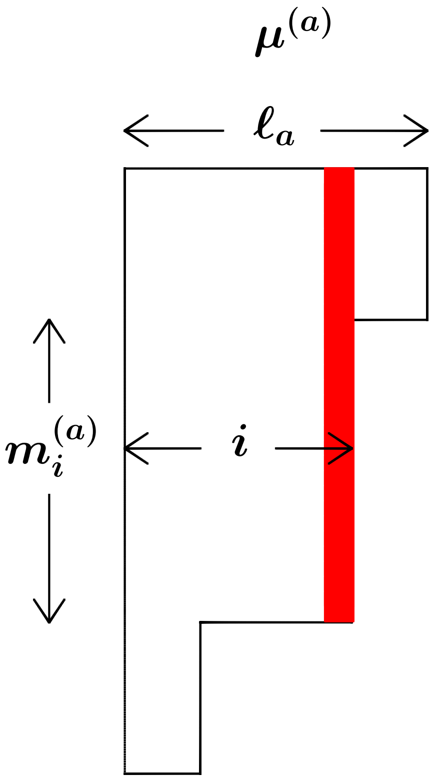


Result. The limit shape of soliton content $(\mu^{(1)}, \dots, \mu^{(n)})$ is given by

$$\eta_i^{(a)} := \lim_{L\to\infty} \frac{1}{L}(\text{Length of the } i \text{ th column of } \mu^{(a)}) = \frac{Q_{i-1}^{(a-1)} Q_i^{(a+1)}}{Q_i^{(a)} Q_{i-1}^{(a)} Q_1^{(1)}}$$

$$\text{width } \ell_a \text{ of } \mu^{(a)} \simeq \frac{\log L}{\log \frac{p_{a-1}}{p_a}} \quad (L \to \infty \text{ if } p_0 > \cdots > p_n)$$

Special case $p_a = \frac{q^a}{1+q+\cdots+q^n} \quad (0 < q \le 1).$



Scaled column length of $\mu^{(a)}$

$$\eta_i^{(a)} = \frac{q^{i+a-1}(1-q)(1-q^a)(1-q^{n+1-a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})}$$

Strings

$$\rho_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} m_i^{(a)} = \frac{q^{i+a-1}(1-q)^2(1-q^a)(1-q^{n+1-a})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

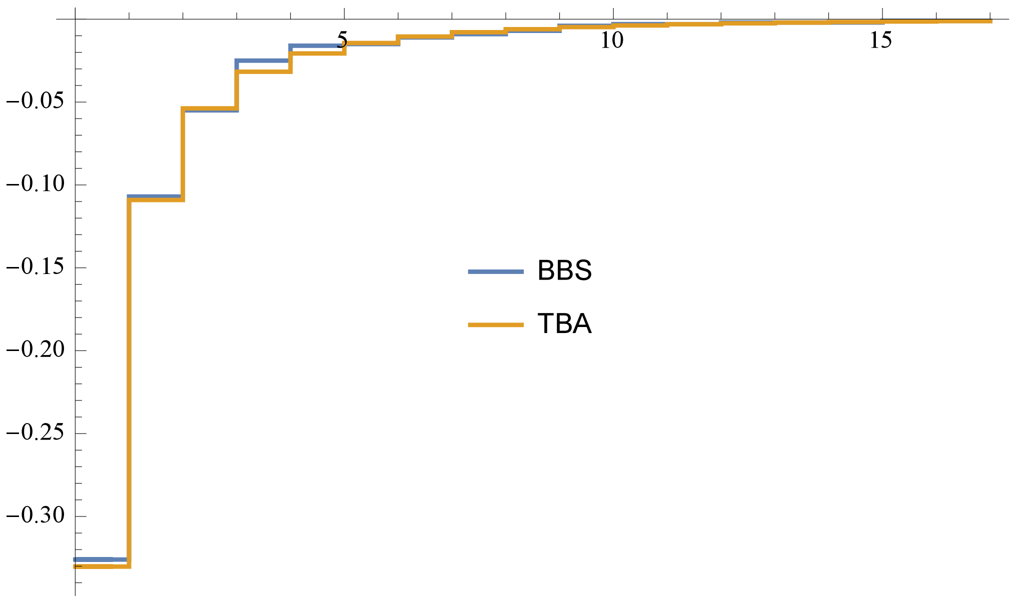
Holes

$$\sigma_i^{(a)} = \lim_{L\to\infty} \frac{1}{L} h_i^{(a)} = \frac{q^{a-1}(1-q)^2(1-q^i)(1-q^{n+i+1})(1+q^{i+a})}{(1-q^{n+1})(1-q^{i+a-1})(1-q^{i+a})(1-q^{i+a+1})}$$

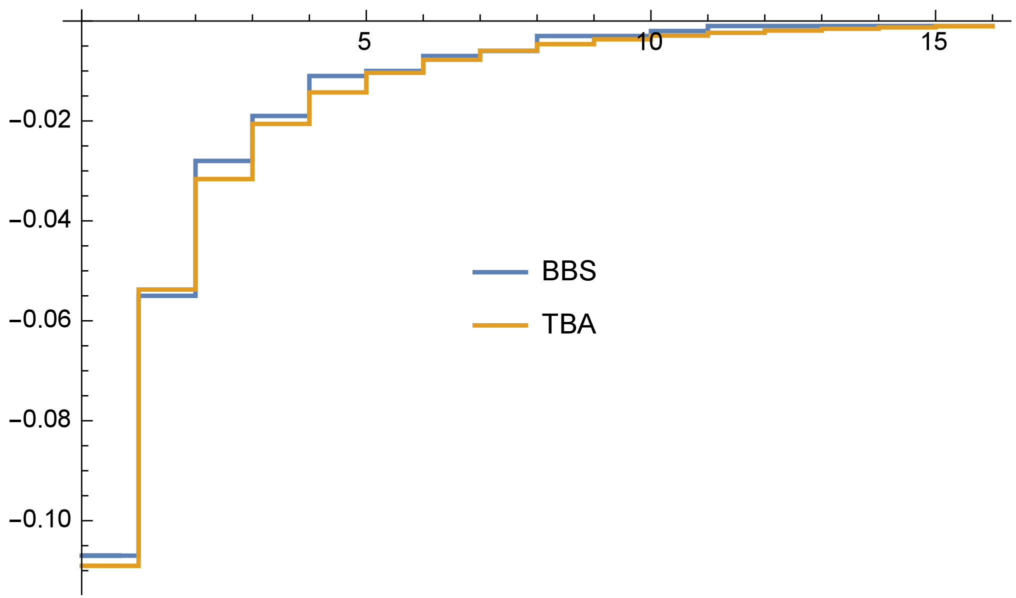
2-color BBS with $L = 1000$ sites with distribution $(p_0, p_1, p_2) = (\frac{7}{18}, \frac{6}{18}, \frac{5}{18})$.

Vertically L^{-1} scaled soliton contents.

$\mu^{(1)}$



$\mu^{(2)}$



Generalized hydrodynamics (GHD) (from here 1-color BBS only) [K-Misguich-Pasquier, '20,21,22]

[Castro, Alvaredo, Doyon, Yoshimura, Bertini, Collura, De Nardis, Fagotti,...]

GGE(β_1, β_∞) : ball density $= \dfrac{a}{1+a}$, soliton density $= \dfrac{a(1-z)}{(1+a)(1-az)}$ (i.i.d : $a = z = q$)

Densities: ρ_i (i-soliton), σ_i (i-hole), $(\rho_i, \sigma_i) = (\rho_i^{(1)}, \sigma_i^{(1)})$ in previous pages

$$\rho_i = \frac{az^{i-1}(1-a)(1-z)^2(1+az^i)}{(1+a)(1-az^{i-1})(1-az^i)(1-az^{i+1})} \qquad \sigma_i = \frac{(1-a)(1+az^i)}{(1+a)(1-az^i)}$$

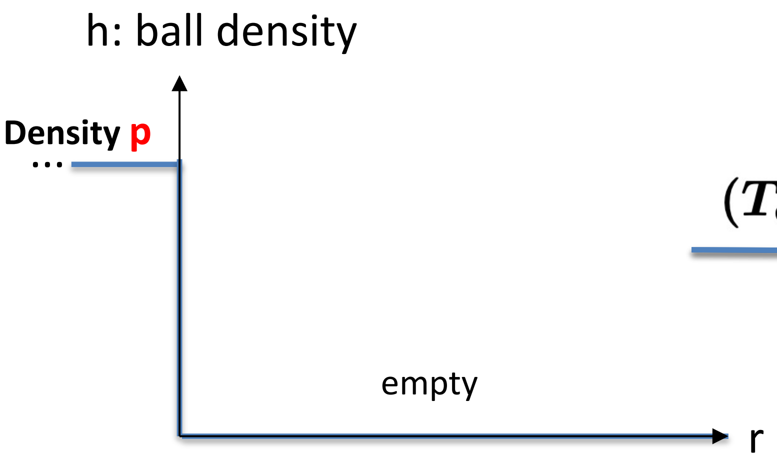
Speed equation: $v_j^{(l)} = \min(j,l) + \sum_{k=1}^{\infty} 2\min(j,k)(v_j^{(l)} - v_k^{(l)})\rho_k$

$(2\min(i,j) = \text{ phase shift})$ [Croydon,Sasada]

$$v_j^{(l)} = \sum_{k=1}^{\min(j,l)} \frac{\sigma_l}{\sigma_{k-1}\sigma_k}, \qquad v_j = \sum_{k=1}^{j} \frac{\sigma_\infty}{\sigma_{k-1}\sigma_k} \qquad v_k^{(l)} = \frac{1+az^l}{1-az^l} v_{\min(k,l)}, \quad v_k = \frac{1+a}{1-a}k - \frac{2a(1+z)(1-z^k)}{(1-a)(1-z)(1+az^k)}$$

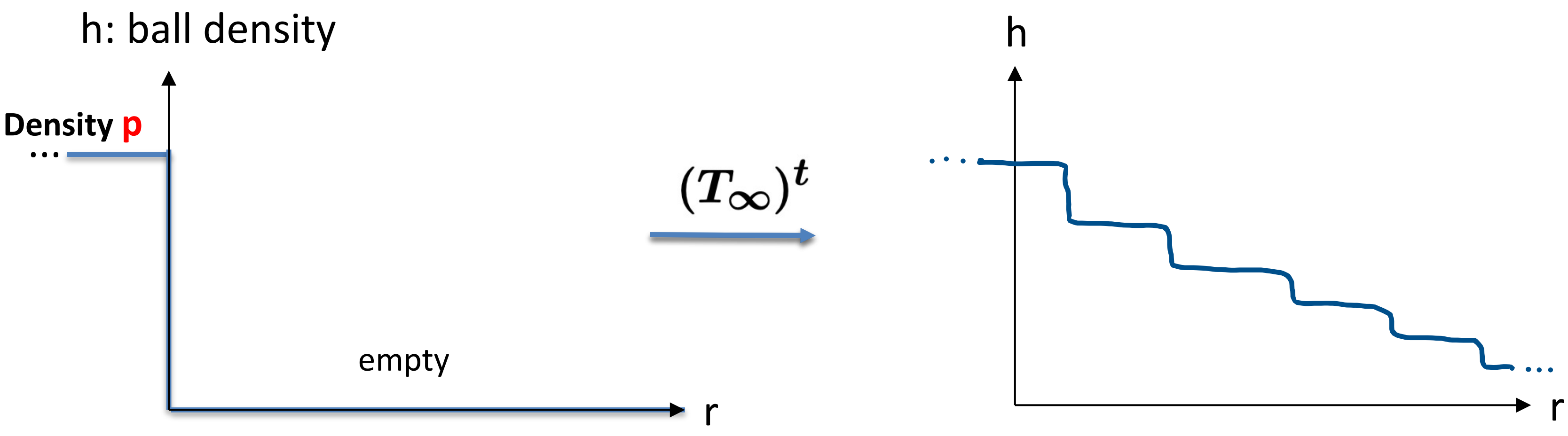
Stationary ball current under $T_l = \sum_{k=1}^{\infty} k\rho_k v_k^{(l)} = \dfrac{a(1+z)}{(1+a)(1-z)}\left(1 - \dfrac{(1-a)z^l}{1-az^l}\right) - \dfrac{laz^l}{1-az^l}$

Density Plateaux emerging from domain wall initial condition

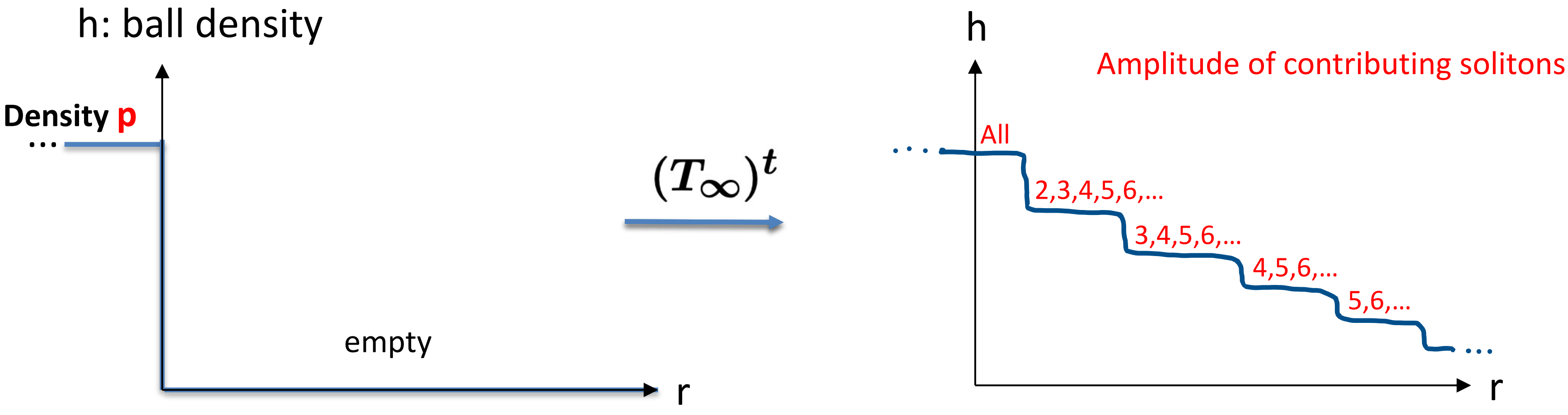


$(T_\infty)^t$

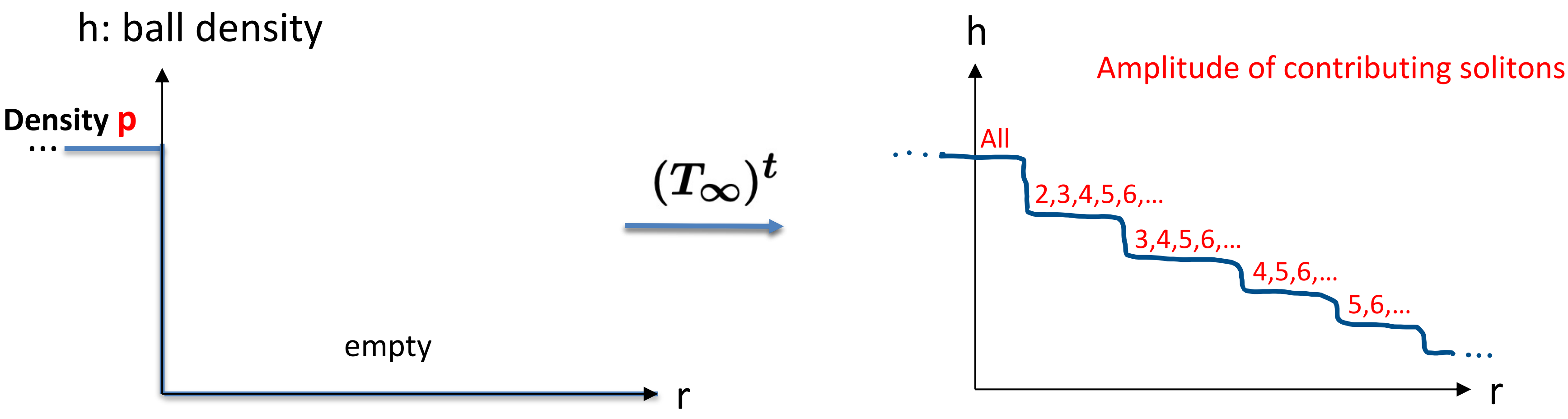
Density Plateaux emerging from domain wall initial condition



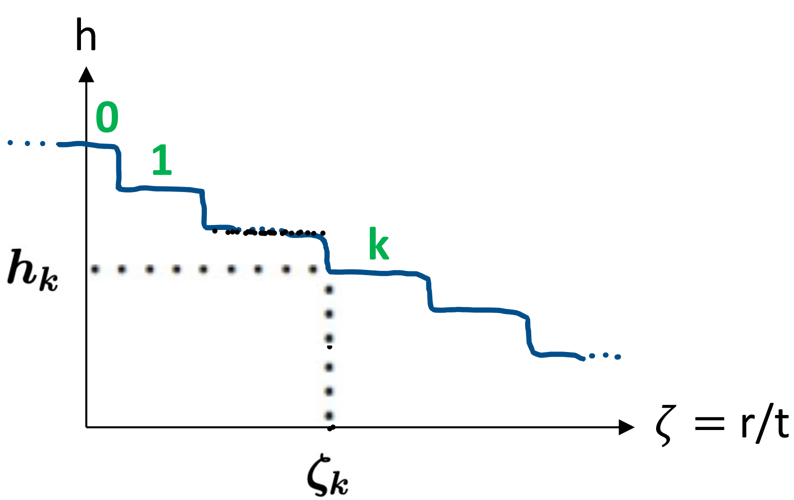
Density Plateaux emerging from domain wall initial condition



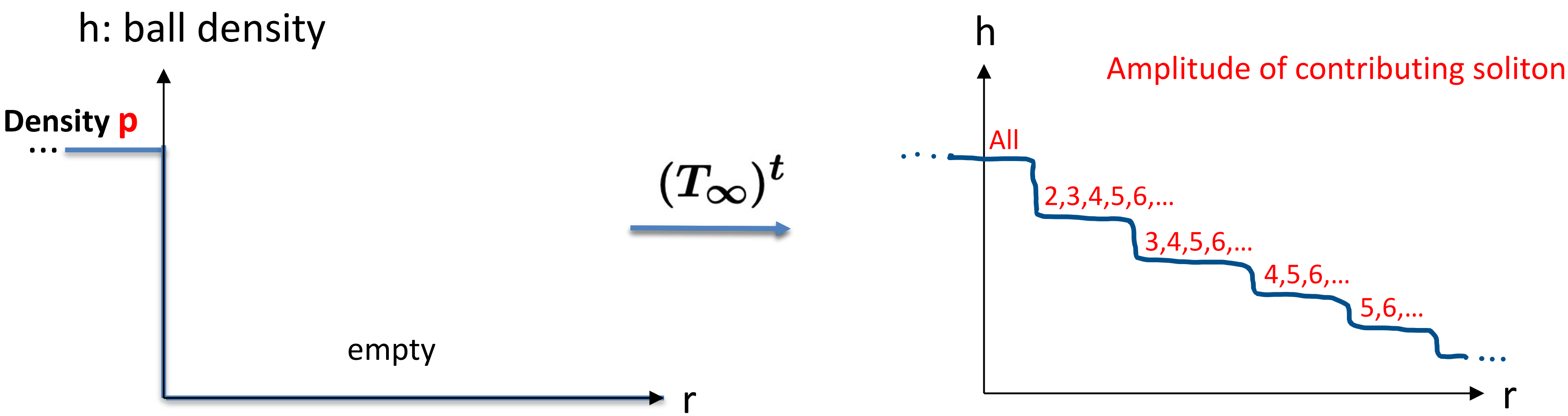
Density Plateaux emerging from domain wall initial condition



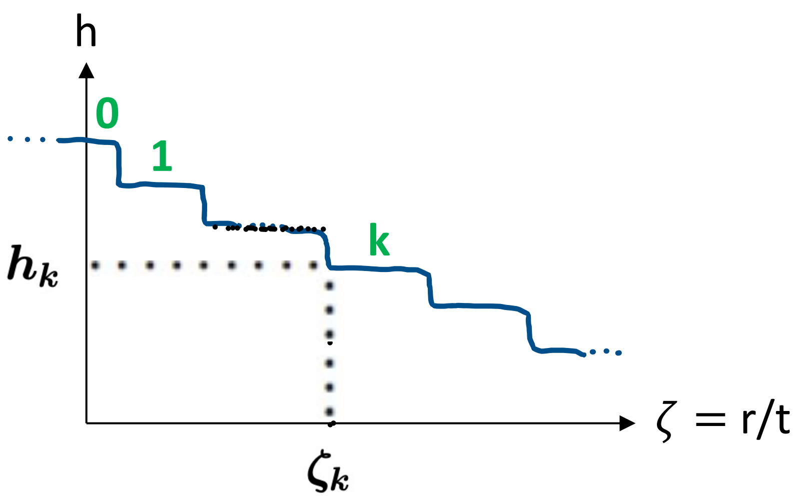
Plateaux grow linearly in time t. The plot against ζ = r/t collapses into a single curve.



Density Plateaux emerging from domain wall initial condition



Plateaux grow linearly in time t. The plot against ζ = r/t collapses into a single curve.



GHD prediction

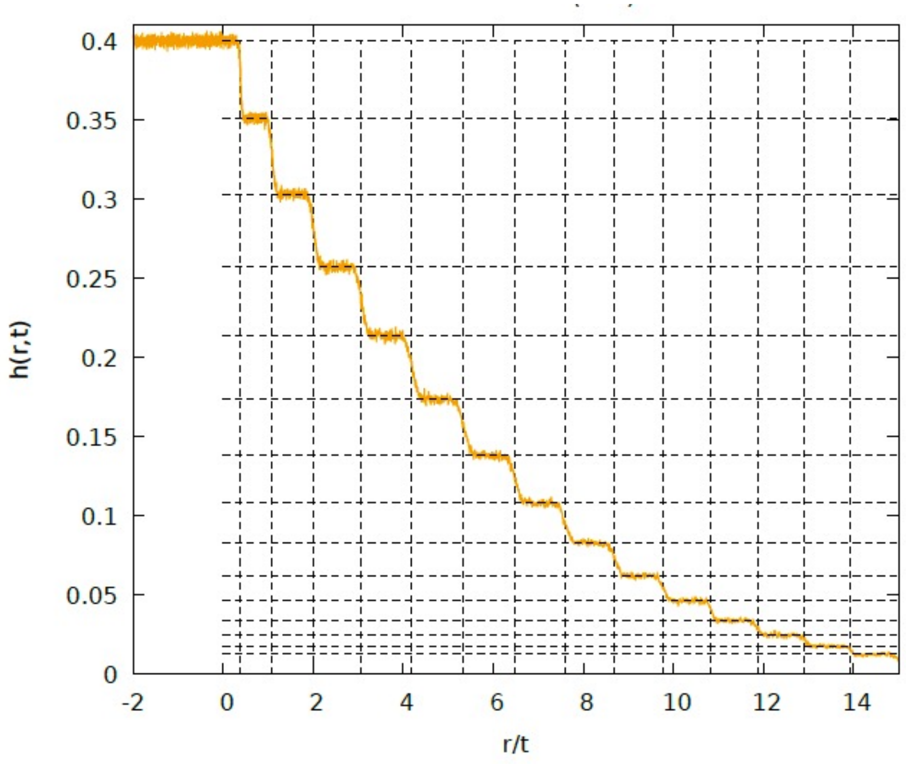
$$h_k = \frac{q^{k+1}(1 - q^{k+2} + k(1-q))}{1 - q^{2k+3} + (2k+1)(1-q)q^{k+1}}$$

$$\zeta_k = \frac{k(1-q^{k+1})}{1+q^{k+1}} \qquad \left(p = \frac{q}{1+q}\right)$$

Simulation with $N_{samples}$ = 50000

(Plots of ball density vs $\zeta = r/t$. Dotted lines are GHD predictions)

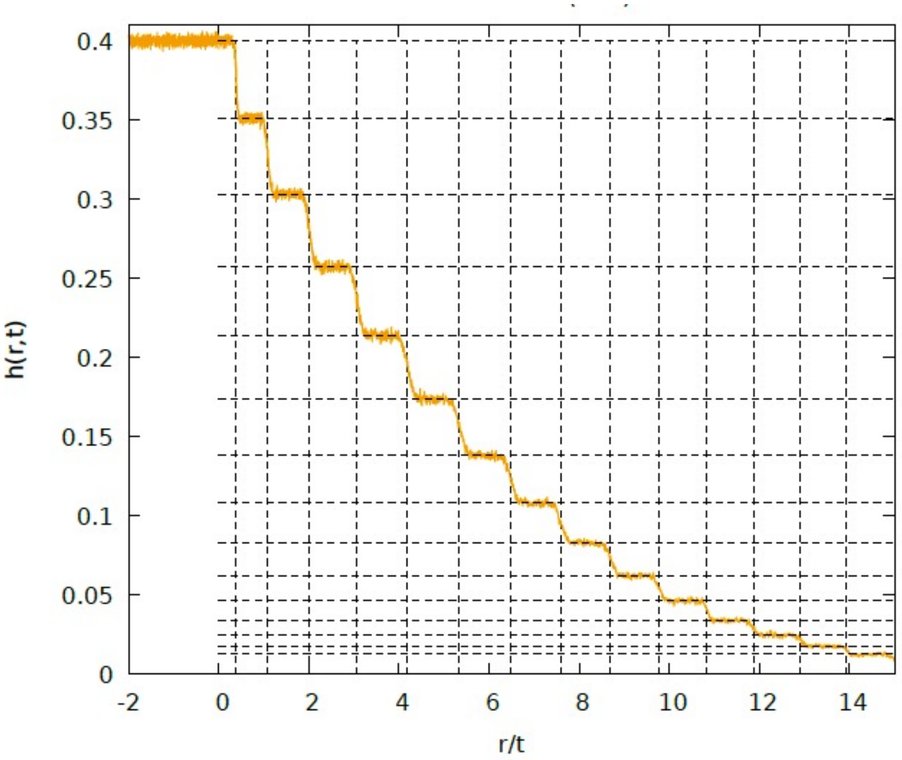
p=0.4, q=0.666…, t=500.



Simulation with $N_{samples}$ = 50000

(Plots of ball density vs $\zeta = r/t$. Dotted lines are GHD predictions)

p=0.4, q=0.666…, t=500.



Actual plateau edges are not strict and exhibit some **broadening**.

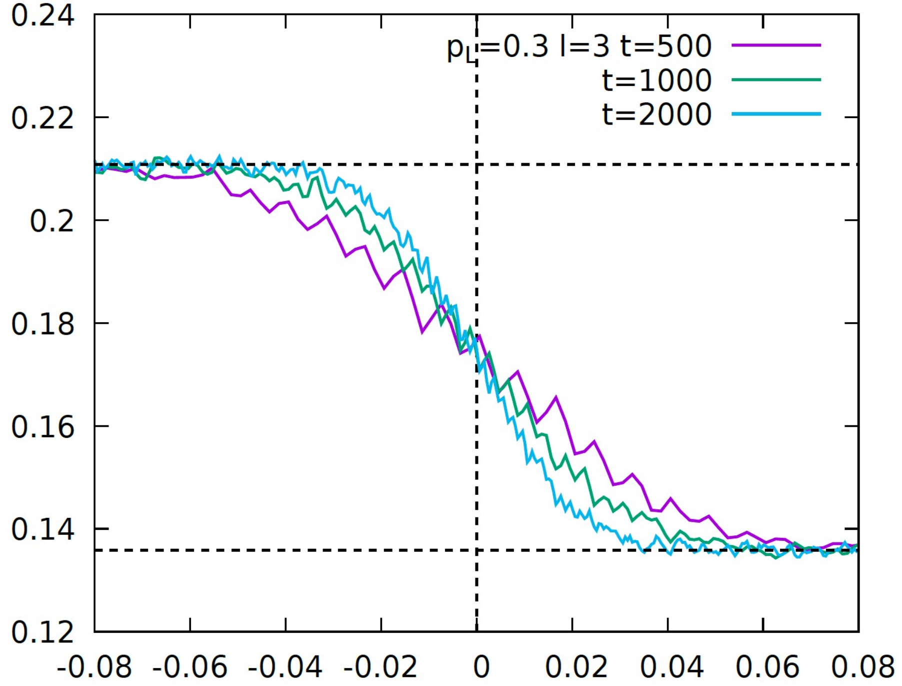
This is due to **diffusive** correction to the **ballistic** picture,
which may be viewed as a finite t effect.

Analytical description of the **diffusive broadening** of plateau edges

Position of plateau edge - ζ

fluctuates over the scale

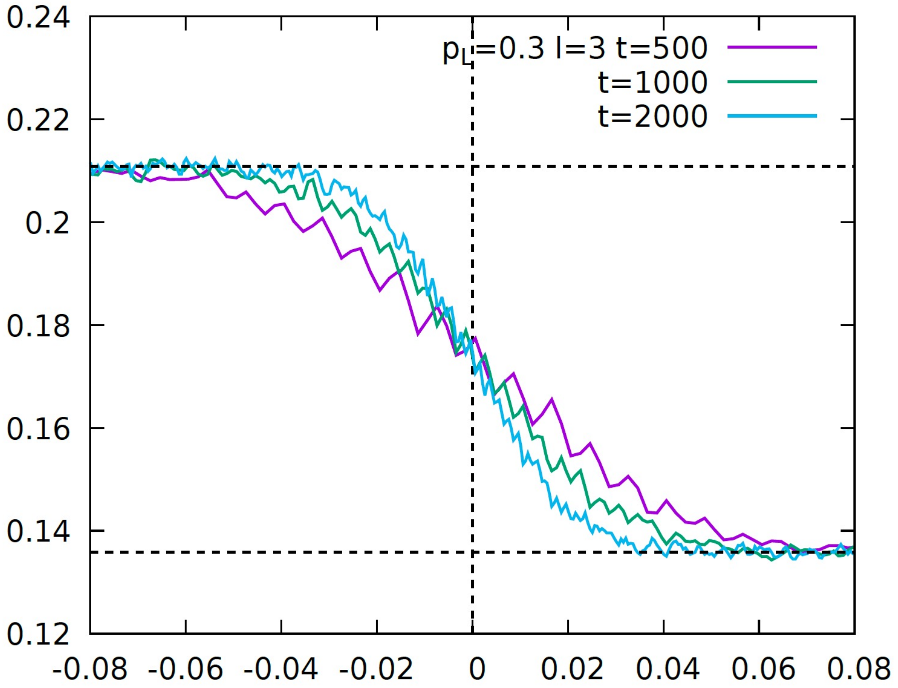
$$\frac{1}{\sqrt{(\mathbf{Diffusion\ const})t}}$$



Analytical description of the **diffusive broadening** of plateau edges

Position of plateau edge - ζ fluctuates over the scale

$$\frac{1}{\sqrt{(\textbf{Diffusion const})t}}$$



$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \mathrm{e}^{-s^2} ds$$

$$\langle \boldsymbol{\rho_j(r,t)} \rangle = \frac{1}{2}\left(\boldsymbol{\rho_j(k-1)} - \boldsymbol{\rho_j(k)}\right) \mathbf{erfc}\left(\frac{\boldsymbol{r - \zeta(k)t}}{\sqrt{\boldsymbol{2t}}\,\boldsymbol{\Sigma_k^{(l)}}}\right) + \boldsymbol{\rho_j(k)}$$

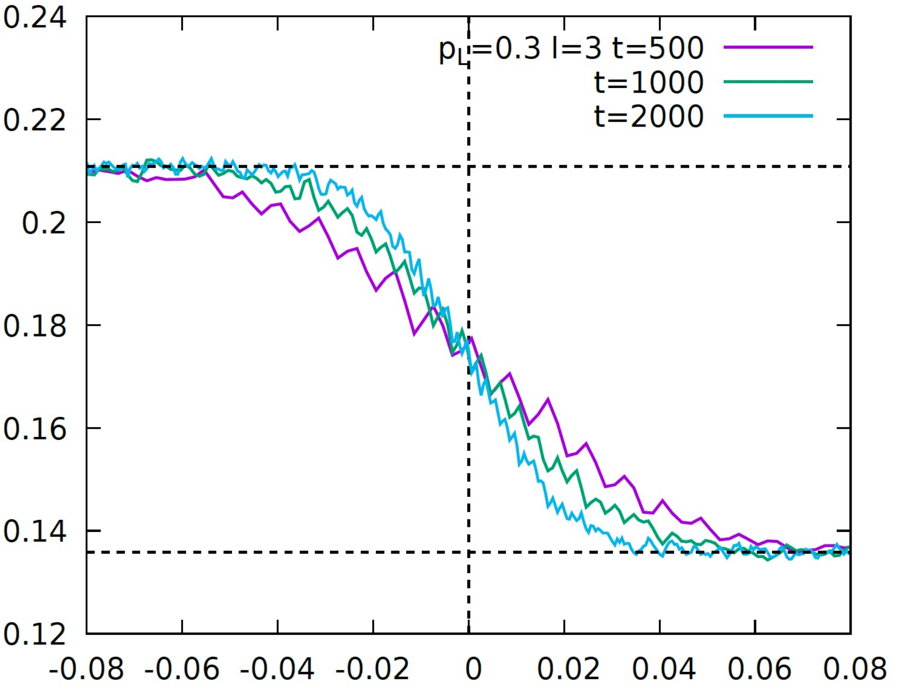
amplitude j-soliton density around the k th plateau edge $r = \zeta(k)t$ under the time evolution T_l.

Analytical description of the **diffusive broadening** of plateau edges

Position of plateau edge - ζ

fluctuates over the scale

$$\frac{1}{\sqrt{(\textbf{Diffusion const})t}}$$



$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty e^{-s^2} ds$$

$$\langle \rho_j(r,t) \rangle = \frac{1}{2} (\rho_j(k-1) - \rho_j(k)) \, \mathrm{erfc}\left(\frac{r - \zeta(k)t}{\sqrt{2t}\, \Sigma_k^{(l)}} \right) + \rho_j(k)$$

amplitude j-soliton density around the k th plateau edge $r = \zeta(k)t$ under the time evolution T_l.

$$(\Sigma_k^{(l)})^2 = \frac{4k^2 q^{k+1}(1-q^{k+1})(1-q^{l-k})(1+q^{l+k+2})}{(1+q^{k+1})^3(1-q^{l+1})^2}$$

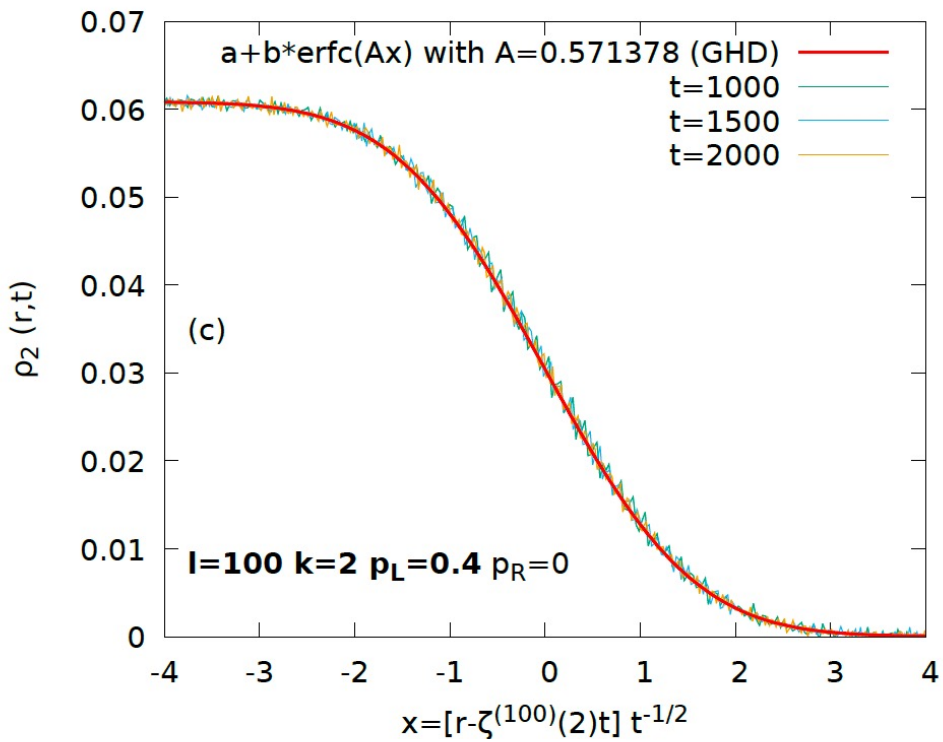
← GHD (Bethe ansatz)

Analytical description of the **diffusive broadening** of plateau edges

Position of plateau edge - ζ

fluctuates over the scale

$$\frac{1}{\sqrt{(\textbf{Diffusion const})t}}$$



$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \mathrm{e}^{-s^2} ds$$

$$\langle \rho_j(r,t) \rangle = \frac{1}{2}\left(\rho_j(k-1) - \rho_j(k)\right) \mathrm{erfc}\left(\frac{r-\zeta(k)t}{\sqrt{2t}\,\Sigma_k^{(l)}}\right) + \rho_j(k)$$

amplitude j-soliton density around the k th plateau edge $r = \zeta(k)t$ under the time evolution T_l.

$$(\Sigma_k^{(l)})^2 = \frac{4k^2 q^{k+1}(1-q^{k+1})(1-q^{l-k})(1+q^{l+k+2})}{(1+q^{k+1})^3(1-q^{l+1})^2}$$

← GHD (Bethe ansatz)

Some Additional Results

- Equation of motion of BBS is "bilinearized" as

$$\boldsymbol{\tau} + \boldsymbol{\tau} = \max(\boldsymbol{\tau} + \boldsymbol{\tau}, \boldsymbol{\tau} + \boldsymbol{\tau})$$

$\boldsymbol{\tau}$ = UD analogue of Hirota tau function
(= corner transfer matrix of BBS)

- Explicit piecewise linear formula for the KKR map

- For Periodic BBS, fundamental cycle $\mathcal{N}$ satisfies

$$\left(\text{Bethe eigenvalue of transfer matrix}|_{q=0}\right)^{\mathcal{N}} = 1$$

- Solution of the initial value problem is given as

Local state $= \boldsymbol{\Theta} - \boldsymbol{\Theta} + \boldsymbol{\Theta} - \boldsymbol{\Theta}$,
$\boldsymbol{\Theta}$ = Tropical (UD analogue of) Riemann theta function